Symmetric vs. Sum Capacity of Rayleigh MAC

or: Probability of Achieving Fairness for Free

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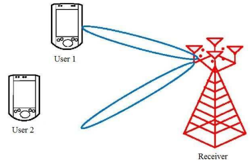
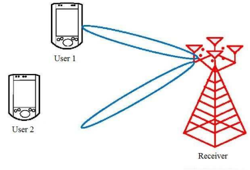
Orthogonal vs. non-orthogonal multiple access



- How to share the medium: issue in both uplink and downlink
- Orthogonal multiple access OMA (e.g., via time/frequency) has been the dominant approach in past cellular comm. generations:
  - FDMA (1G)
  - TDMA (2G)
  - Synchronous CDMA (2G & 3G)
  - OFDMA (4G)
- Any reason to consider other methods?

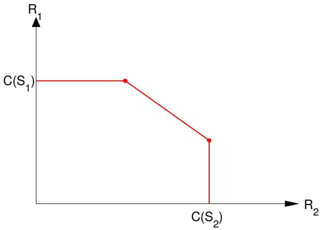
Uplink



Consider for simplicity two-user MAC (extension to N-user is easy), each terminal has single antenna

$$y = h_1 x_1 + h_2 x_2 + n$$

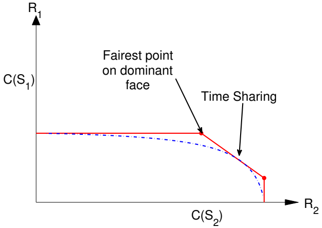
Both users have power P and $\sigma_n^2 = 1$



Uplink

Traditional approach

- Coordinate so that users don't collide - 1 active user per DoF
- Users that overlap (overloaded scenario) are treated as noise



Rate Region of OMA

For two uses, with $0 \leq \alpha \leq 1$, can achieve

$$R_1(\alpha) = \alpha \log\left(1 + \frac{P}{\alpha}|h_1|^2\right) \;;\; R_2(\alpha) = (1-\alpha)\log\left(1 + \frac{P}{1-\alpha}|h_2|^2\right)$$

Uplink

Traditional approach

- Upside: OMA is throughput optimal
- Downside:
  - Fairness issue- throughput optimality does not hold under individual rate requirements
  - Coordination may be a major issue (grant-free transmission) - overloaded system

Non orthogonal multiple access (NOMA)

- Capacity-achieving practical NOMA schemes
  - Superposition coding + SIC + time sharing (power-domain NOMA)
  - Rate splitting
- Upside: Both achieve capacity region
- Downside: Both require user coordination...

Uncoordinated transmission

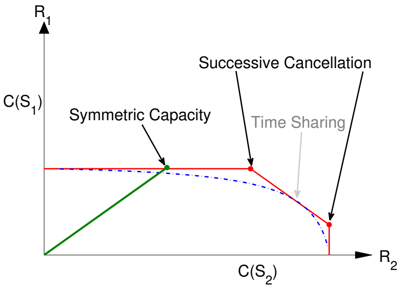
Definition

- Multiple users transmit simultaneously (occupying same DoFs) with equal power and equal rate
- No *ordering coordination*: time sharing and rate splitting not applicable...

Why is it important?

- Next gen wireless communication:
  - Is going to be very crowded
  - Latency is major issue for some applications
  - Ad-hoc networks
- Coordination
  - May entail large overheads
  - May result in increased latency

Multiple access capacity region: another look



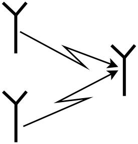
- $C_{\mathrm{sym}} = \max \min(R_1, R_2)$ where minimization over all $(R_1, R_2) \in$ capacity region
- In theory: Symmetric capacity $\Longrightarrow$ No coordination
- In practice...

The topic of this talk

- What is the probability that symmetric capacity equals sum capacity?
  - Less formally - how much do you pay for striving for fairness?
- Practical multi-user detector for the symmetric capacity?

Channel model

- MAC: $$y = \sum_{i=1}^{N} h_i x_i + z$$



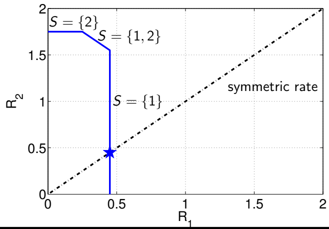
- CSI at Rx
- Equal average transmission power per antenna: $P = 1$
- $z \sim \mathcal{CN}(0,1)$
- $h_i \sim \sqrt{\mathrm{SNR}} \cdot \mathcal{CN}(0,1)$ and i.i.d. (symmetric setting)

Definitions

- Sum capacity: $C_{\text{sum}} = \log\left(1 + \sum |h_i|^2\right)$
- Capacity region (set of constraints):

$$C(\mathbf{h}) = \sum_{i \in S} R_i \leq \log\left(1 + \sum_{i \in S} |h_i|^2\right), \; S \subseteq \{1, \ldots, N\}$$

- Symmetric capacity:
  - $C_{\text{sym}} = \max_{\mathbf{R} \in C(\mathbf{h})} \min(R_1, \ldots, R_N) = \min_{S \subseteq \{1,\ldots,N\}} \frac{1}{|S|} \log\left(1 + \sum_{i \in S} |h_i|^2\right)$
  - $C_{\Sigma-\text{sym}} = N \cdot C_{\text{sym}}$



Need to analyze the bottleneck !

Symmetric vs. sum capacity

- $C_{\Sigma-\mathrm{sym}} = C_{\mathrm{sum}} \Rightarrow$ fairness comes for free!
- But what are the chances of that happening?
- **Q1**: What is $P\left(C_{\Sigma-\mathrm{sym}} < R \middle| C_{\mathrm{sum}} = c_{\mathrm{sum}}\right)$?
- **Q2**: Probability that $C_{\Sigma-\mathrm{sym}} = C_{\mathrm{sum}} = \log\left(1 + \sum_{i=1}^{N} |h_i|^2\right)$ is?
- We analyze the probabilities given C_{sum}
- Let's start with **Q2** on a concrete example: $C_{\mathrm{sum}} = 2$

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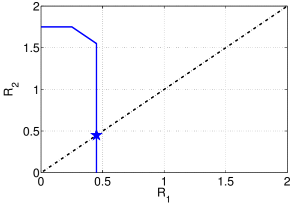

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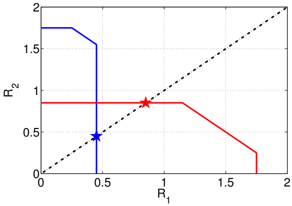

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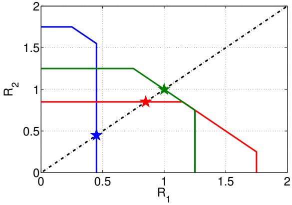

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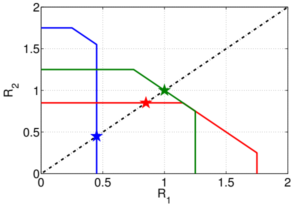

Well, there are three faces so... 1/3?

Symmetric vs. sum capacity

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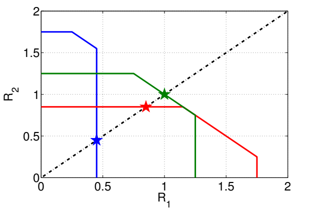
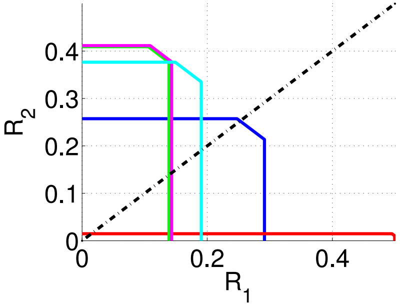
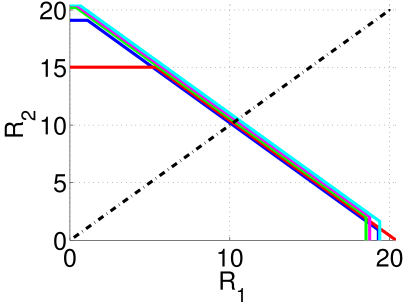


Correct answer, explanation can be improved...

What is known

- $C_{\Sigma-\text{sym}} \leq C_{\text{sum}}$
- (Implicitly from the MAC-DMT): SNR $\to \infty \;\Rightarrow\; C_{\Sigma-\text{sym}} \overset{w.h.p.}{\to} C_{\text{sum}}$





- Our goal: analyze the (finite SNR) distribution of $C_{\Sigma-\text{sym}}$ given C_{sum}

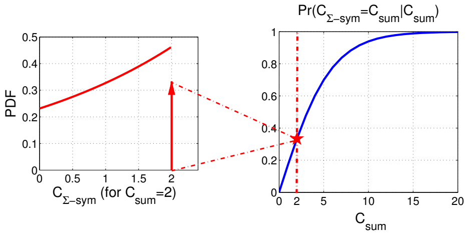
Bottom line - two-user Rayleigh MAC

Theorem 1

For a 1×2 Rayleigh MAC with sum capacity C_{sum}:

$$P(C_{\Sigma-\text{sym}} < R | C_{\text{sum}}) = 2 \cdot \frac{2^{R/2} - 1}{2^{C_{\text{sum}}} - 1}; \; 0 \le R \le \boldsymbol{C_{\text{sum}}}$$

$$\begin{aligned} P(C_{\Sigma-\text{sym}} = C_{\text{sum}} | C_{\text{sum}}) &= 1 - P(C_{\Sigma-\text{sym}} < C_{\text{sum}} | C_{\text{sum}}) \\ &= 1 - 2 \cdot \frac{2^{C_{\text{sum}}/2} - 1}{2^{C_{\text{sum}}} - 1} \end{aligned}$$



Symmetric vs. sum capacity

More than two users

Inner and outer bounds

But why condition on C_{sum}?

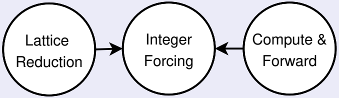
- Elegant expressions...
- Rayleigh (open-loop) outage probability
  - All users (when they are active) transmit at a common target rate R_t
  - Outage probability is then given by $\mathbb{E}_{C_{\mathrm{sum}}}[P(C_{\Sigma-\mathrm{sym}} < NR_t | C_{\mathrm{sum}})]$
- Simple MAC transmission protocol
  - Receiver learns channel gains of active users
  - Calculates $C_{\Sigma-\mathrm{sym}}$ and notifies transmitters to each transmit at rate R/N where $R < C_{\Sigma-\mathrm{sym}}$:
    - Trivial rate allocation
    - Minimal feedback

Implementation

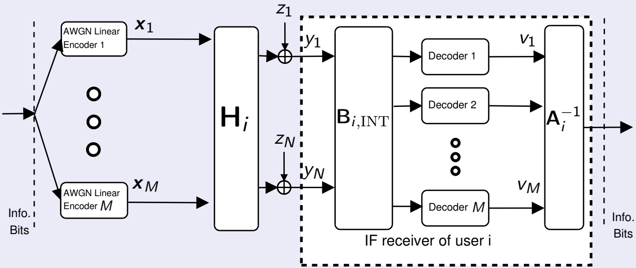
But how do we achieve the symmetric capacity in practice?

Candidate MUD for equal rate transmission: integer forcing

- Equalization scheme introduced by Zhan '10, et. al.



- Idea: Decode linear combination of messages $\Longrightarrow$ Invert



How close IF to symmetric capacity?

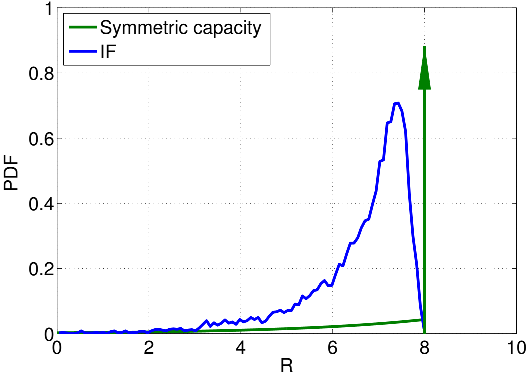


Figure: PDF of achievable rate given $C_{\mathrm{sum}} = 8$

- Compared to the symmetric capacity, there's room for improvement...
- But we can do better!

Extensions and improvements

- First improvement
  - IF can be used in conjunction (hybrid operation) with SIC
  - For two-user MAC, when single-user constraint is the bottleneck, then symmetric capacity can be achieved with SIC
- Second improvement (lesson from MAC-DMT):
  - When linear codes are used (uncoded QAM in original context...) - **need to use space-time coding**
    (Lu, Hollanti, Vehkalahti, Lahtonen ('11))
  - Note: In case of a single transmit antenna, the transformation mixes symbols from different time instances

Extensions and improvements

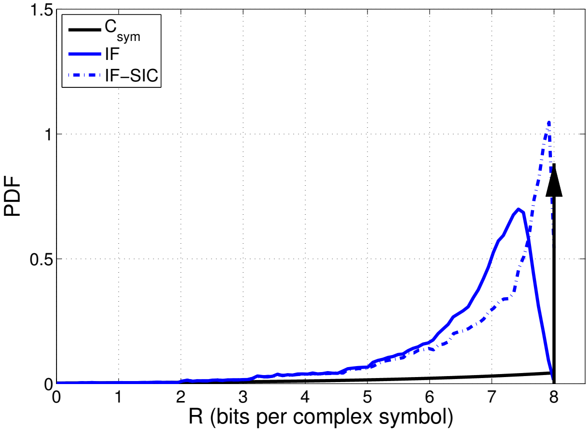


Figure: Symmetric capacity vs. IF for a two-user i.i.d. Rayleigh fading MAC with $C_{\mathrm{sum}} = 8$.

Extensions and improvements

- We demonstrate using precoding suggested by Badr & Belfiore ('08)

$$\begin{bmatrix} y(t=1) \\ y(t=2) \end{bmatrix} = \mathbf{P}_1 \begin{bmatrix} x_1(t=1) \\ x_1(t=2) \end{bmatrix} + \mathbf{P}_2 \begin{bmatrix} x_2(t=1) \\ x_2(t=2) \end{bmatrix} + \begin{bmatrix} n(t=1) \\ n(t=2) \end{bmatrix}$$

where

$$\mathbf{P}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \alpha\phi \\ \bar{\alpha} & \bar{\alpha}\bar{\phi} \end{bmatrix}, \qquad \mathbf{P}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} j\alpha & j\alpha\phi \\ \bar{\alpha} & \bar{\alpha}\bar{\phi} \end{bmatrix}$$

and

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \bar{\phi} = \frac{1-\sqrt{5}}{2}$$

$$\alpha = 1 + j - j\phi, \qquad \bar{\alpha} = 1 + j - j\bar{\phi}.$$

Extensions and improvements

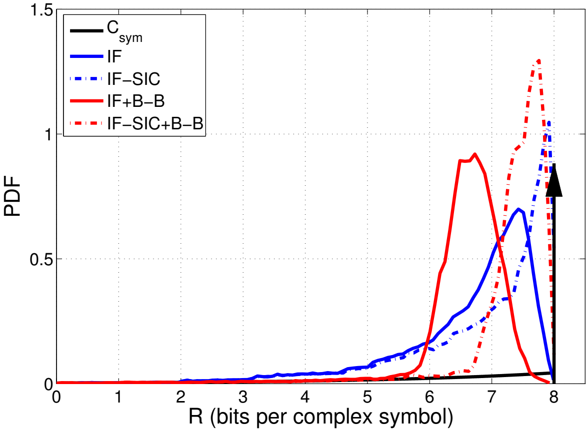


Figure: Symmetric capacity vs. IF for a two-user i.i.d. Rayleigh fading MAC with $C_{\text{sum}} = 8$.

Summary and outlook

Questions raised in this talk

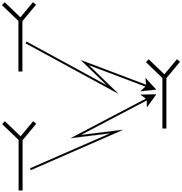
- What is the probability that symmetric capacity equals sum capacity?
- Practical multi-user detector for the symmetric capacity?

Outlook

- Can the bounds for the N-user MAC can be further tightened?
- Can the performance of IF can be further improved by more sophisticated space-time codes?
- Other practical schemes which are closer to the symmetric capacity?

Thank you for your attention

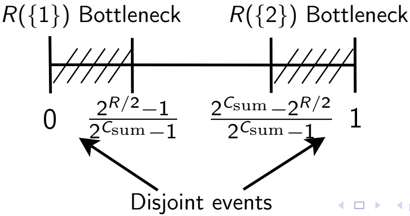
Sketch of Proof: Two-User Rayleigh MAC



- $h_i \sim \mathcal{CN}(0, \text{SNR})$ and i.i.d $\Rightarrow$ $|h_i|^2 \sim exp(\text{SNR})$
- Normalize: $u_i = \frac{1}{\sqrt{2^{C_{\text{sum}}} - 1}} h_i$
- **Given** C_{sum}
  - $|u_1|^2 + |u_2|^2 = 1 \Rightarrow$ zero-sum game
  - $|u_i|^2$ given $|u_1|^2 + |u_2|^2 = 1$ is uniformly distributed over $[0, 1]$ (conditioning property of Poisson process)

Sketch of Proof: Two-User Rayleigh MAC

- $C_{\Sigma-\text{sym}} = N \min_{S \subseteq \{1,\ldots,N\}} R(\{S\}) = N \min_{S \subseteq \{1,\ldots,N\}} \frac{1}{|S|} \log \left(1 + \sum_{i \in S} |h_i|^2 \right)$
- Two users, given C_{sum}: $C_{\Sigma-\text{sym}} = \min\left(2R(\{1\}), 2R(\{2\}), C_{\text{sum}}\right)$
- $R(\{i\}) = \log\left(1 + |u_i|^2 (2^{C_{\text{sum}}} - 1)\right)$
- $P\left(C_{\Sigma-\text{sym}} < R | C_{sum}\right) =$
  $P\left(|u_1|^2 < \frac{2^{R/2}-1}{2^{C_{\text{sum}}}-1}\right) + P\left(|u_1|^2 > \frac{2^{C_{\text{sum}}}-2^{R/2}}{2^{C_{\text{sum}}}-1}\right)$
- $\Rightarrow \; P\left(C_{\Sigma-\text{sym}} < R | C_{sum}\right) = 2\frac{2^{R/2}-1}{2^{C_{\text{sum}}}-1}$



General N: The Bottleneck

- When $N > 2$:
  - There are more possible bottlenecks to check (but remember the DMT moral...)
  - Need to analyze

$$P\left(R(\{S\}) < R | C_{\mathrm{sum}}\right) =$$
$$P\left(\frac{|S|}{N}\log\left(1 + (2^{C_{\mathrm{sum}}} - 1)\sum_{i \in S}|u_i|^2\right) < R \Big| \sum |u_i|^2 = 1\right)$$

  - Possible bottlenecks $\{S\}$ are no longer disjoint
- Tool for analysis
  - Given C_{sum}, u_i can be viewed as elements from a row taken from a unitary matrix drawn from the CUE (Haar measure)
  - Edelman 05' - Singular value distribution of a truncated unitary matrix (eigenvalues have Jacobi/MANOVA distribution)
- $\Rightarrow$ lower and upper bounds

General N: The Bottleneck

Theorem 2 - distribution of a specific set

For a $1 \times N$ Rayleigh MAC with sum capacity C_{sum}, the outage probability for a set $S \subseteq \{1, 2, \dots, N\}$ is

$$P\left(R(\{S\}) < R | C_{\text{sum}}\right) =$$

$$P\left(\frac{|S|}{N} \log\left(1 + (2^{C_{\text{sum}}} - 1) \sum_{i \in S} |u_i|^2\right) < R\Big] \sum |u_i|^2 = 1\right) =$$

$$\frac{\mathcal{B}(\frac{2^{R|S|/N}-1}{2^{C_{\text{sum}}}-1}; |S|, N - |S|)}{\mathcal{B}(1; |S|, N - |S|)}$$

where $0 \leq R \leq C_{\text{sum}}$ and $\mathcal{B}(x; a, b) = \int_0^x u^{a-1}(1-u)^{b-1} du$ is the incomplete beta function.

General N: The Bottleneck

- $P\left(C_{\Sigma-\mathrm{sym}} < R | C_{\mathrm{sum}}\right) = P\left(\min_{S\subseteq\{1,2,\ldots,N\}} R(\{S\}) < R | C_{\mathrm{sum}}\right)$
- All sets with the same cardinality have the same outage probability
- $P_{\mathrm{out}}(k, R) \triangleq P(R(\{|S| = k\} < R | C_{\mathrm{sum}})$
- Union bound can be used to bound overall probability

Theorem 3 - lower and upper bound for N Rayleigh MAC

For a $1 \times N$ Rayleigh MAC with sum capacity C_{sum}, the outage probability can be bounded as

$$\max_k P_{\mathrm{out}}(k, R) \;\leq\; P\left(C_{\Sigma-\mathrm{sym}} < R | C_{\mathrm{sum}}\right) \;\leq\; \sum_{k=1}^{N} \binom{N}{k} P_{\mathrm{out}}(k, R)$$

Upper and Lower Bounds

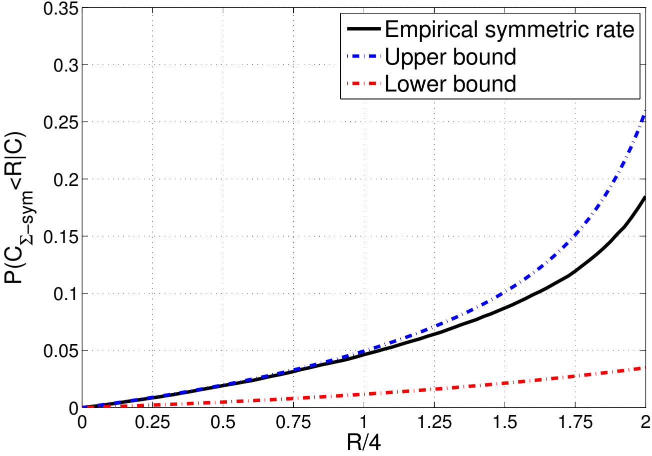


Figure: Bounds vs. Empirical error probability for 1×4 channel with $C_{\text{sum}}/4 = 2$